COUPLED, AMPLITUDE EQUATIONS FOR RAYLEIGH-BÉNARD CONVECTION WITH HEAT SOURCE

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Abstract: A local nonlinear stability analysis using a five-mode expansion is performed in arriving at the coupled amplitude equations for Rayleigh-Bénard convection with heat source. The real amplitudes are determined numerically using the Runge-Kutta-Fehlberg45 method and these are in turn used in estimating the heat transport. The influences of heat source and sink on heat transport are considered and it is found that the effect of heat source is to enhance the heat transport and that of sink is to diminish the same. The parameters influence on heat transport is found to be more pronounced at small time than at long times.

***Keywords**:* Rayleigh- Bénard convection, Heat transport, Heat source(sink), generalized Lorenz model, coupled Ginzburg-Landau model.

Introduction

The problem of convection in Newtonian fluids is of relevance in many fields of applications. There have been a number of studies in the last two decades on the problem of thermal convection in a horizontal layer of fluid in the presence of internal heat sources. Theoretical studies such as those by Roberts (1967), Thirlby (1970), McKenzie et al. (1974), Tveitereid and Palm (1976) and Clever (1977) were all based on the assumption that the internal heat source Q is uniform. Q is not necessarily uniform in many practical problems with applications in nature or in engineering. The effect of non-uniform heat source on onset of convection in viscoelastic liquids has recently been investigated by Siddheshwar et al. (2010).

It is with the motive of understanding heat transport, we have made a weakly non-linear analysis of Bénard-Darcy convection in the presence of heat source using Landau--Ginzburg model. Landau-Ginzburg model is one of the most studied equations in applied mathematics. It describes a vast array of phenomena including non-linear waves, second-order phase transitions, Rayleigh-–Bénard convection and superconductivity (Siddheshwar 2010). Ginzburg-Landau equation is one of the most studied equations due to its application in explaining non-linear phenomenon. The objective of the paper is to present a coupled system of two real Ginzburg-Landau equations for an analytical study of Rayleigh–Bénard convection with heat source.

Mathematical Formulation

The governing equations describing the Rayleigh- Bénard instability situation of a constant viscosity Newtonian fluid in a porous medium with temperature dependent heat source in the dimensionless form are:

$$(1/\mathrm{Pr})\left(\partial\left(\nabla^2\Psi\right)/\partial\tau\right)=\nabla^4\Psi+R_E\left(\partial\Theta/\partial X\right)-(1/\mathrm{Pr})J\left(\Psi,\nabla^2\Psi\right), \quad (1)$$

$$\partial\Theta/\partial\tau= -\left(\partial\Psi/\partial X\right)\left(df/dz\right)+\nabla^2\Theta+R_I\Theta-J\left(\Psi,\Theta\right), \quad (2)$$

where J is the Jacobian.

The non-dimensional parameters appearing in Eqs. (1) – (2) are defined below:

$R_E = \beta g \Delta T d^3 / \nu \chi_v$ (External Rayleigh number), $\Pr = \nu/\chi$ (Prandtl number) and $R_I = Qd^2/\chi_v$ (Internal Rayleigh number).

Eqs. (1) – (2) are solved using the boundary conditions

$$\Psi = \partial^2 \Psi / \partial Z^2 = \Theta = 0, \quad \text{at } Z = 0, 1. \tag{3}$$

Local Non-Linear Stability Analysis Using Coupled Ginzburg-Landau Equations

Substituting the five- mode truncated Fourier series expansion given by

$$\left.\begin{aligned} \Psi(X,Z,\tau) &= \left(\sqrt{2}/\pi^2 a_c\right)\left(k^2 - R_I\right)\left[A_1(\tau)\sin\pi\alpha_c X - A_2(\tau)\cos\pi\alpha_c X\right]\sin\pi Z, \\ \Theta(X,Z,\tau) &= \frac{4\pi}{r_E\left(4\pi^2 - R_I\right)}\left[\begin{pmatrix} \sqrt{2}B_1(\tau)\cos\pi\alpha_c X \\ +\sqrt{2}B_2(\tau)\sin\pi\alpha_c X \end{pmatrix}\sin\pi Z - C(\tau)\sin 2\pi Z\right] \end{aligned}\right\}, \tag{4}$$

into Eqs. (1)-(2) and following standard orthogonolization procedure for the Galerkin expansion, we obtain the following nonlinear autonomous system (generalized Lorenz model, Sparrow, 1981)

Eliminating B_1, B_2 and C from the Lorenz model, we get

$$dA_1/d\tau = \left(a\Pr^*/\left\{1 + a\Pr^*\right\}\right)\left(\left\{k^2 - R_I\right\}/b\right)\left[b\left(r_E - 1\right)A_1 - A_1^3 - A_1 A_2^2\right]. \tag{5}$$

where $\Pr^* = \Pr/\left\{k^2 - R_I\right\}$.

$$dA_2/d\tau = \left(\left\{a\Pr^*\right\}/\left\{1 + a\Pr^*\right\}\right)\left(\left\{k^2 - R_I\right\}/b\right)\left[b\left(r_E - 1\right)A_2 - A_2^3 - A_2 A_1^2\right]. \tag{6}$$

Equations (5) and (6) are obviously the coupled Ginzburg–Landau model for nonlinear convection in a Newtonian liquid with heat source.

Combining Eq.(5) and (6) using a new variable,

$$A = A_1 + iA_2 \tag{7}$$

we get $$dA/d\tau = \left(\left\{a\Pr^*\right\}/\left\{1 + a\Pr^*\right\}\right)\left(\left\{k^2 - R_I\right\}/b\right)\left[b\left(r_E - 1\right)A - |A|^2 A\right]. \tag{8}$$

Writing A in the phase-amplitude form, we get,

$$A = |A| e^{i\Phi} \tag{9}$$

Now substituting Eq. (9) in Eq. (8), we get the following equation for the amplitude $|A|$:

$$d|A|/d\tau = \left(\left\{a\Pr^*\right\}/\left\{1 + a\Pr^*\right\}\right)\left(\left\{k^2 - R_I\right\}/b\right)\left[b\left(r_E - 1\right)|A| - |A|^3\right]. \tag{10}$$

Equation (10) is a Bernoulli equation in $|A|$ which can be solved in a closed form subject to an initial condition A(0).

In the next section we quantify the heat transport at the lower boundary in terms of the Nusselt number.

Heat Transport at the Lower Boundary

The horizontally-averaged Nusselt number, Nu, for the stationary mode of convection (the preferred mode in this problem) is given by

$$Nu(\tau)=\left[1+2\Big/\left\{r_E\left(1-R_I/4\pi^2\right)\right\}\right]\left(\tan\sqrt{R_I}\Big/\sqrt{R_I}\right)C(\tau). \quad (11)$$

The two terms on the right side of Eq. (11) characterizes the conductive and convective contribution to the heat transport respectively on a length scale *d*.

Results and Discussions

To study the buoyancy induced convection, the value of R_I is assumed to be small enough not to induce convection by itself. The results are depicted in the following figure which is a plot of Nusselt number, *Nu*, versus time, τ, for different values of R_I for both heat source and sink. From the figure it is clear that heat source $(R_I > 0)$ enhances the heat transport and heat sink $(R_I < 0)$ diminishes it.

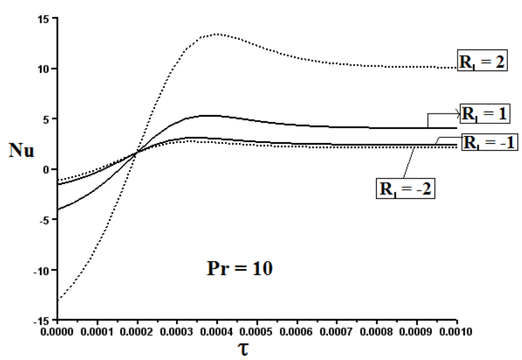

Fig. Plot of Nusselt number, *Nu*, versus time, τ, for different values of R_I.

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