64th CONGRESS OF ISTAM

Section Code: FM-13 (Mathematical Modelling)

Hypersingular integral equation approach for impermeable plate in deep water

Panduranga K*, Santanu Koley, Dipak K Satpathi

Department of Mathematics, Birla Institute of Technology & Science, Pilani– Hyderabad Campus.

*E-mail: panduranga3006@gmail.com

1. Abstract

In this paper, we studied the interaction of water waves with an impermeable vertical plate in deep water. To convert the boundary value problem into hypersingular integral equations, appropriate Green's function is derived and used. This derived hypersingular integral equation will be handled for solution using suitable numerical method. Various results associated with wave-structure interaction problem such as the reflection and transmission coefficients, wave forces acting on the structure will be obtained and discussed in details.

2. Literature survey

Hypersingular integral equations based techniques are widely used to handle a number of problems in wave-structure interactions. Kaya and Erdogan (1987) derived some results in integral equations with strong kernel using Hadamard finite part integrals. Martin and Rizzo (1989) studied the hypersingular integral equation method for three-dimensional crack problems. Krishnasamy et al. (1990) used appropriate solution techniques to handle a number of hypersingular integral equations which appears while analyzing water waves interaction with thin plates and cracks. Parsons and Martin (1992) introduced the concept of hypersingular integral equations to analyze the scattering of water waves by submerged curved and flat plates floating in infinite and finite water depths by taking linear water wave theory into consideration. Later, Kuzenetsov et al. (1998) applied the same procedure to study the existence of trapped modes. Mandal and Gayan (2002) studied the water wave scattering by two symmetrical circular arc plates. Recently, Gayen and Mondal (2014), Chakraborty et al. (2016) and Gupta and Gayen (2019) have used hypersingular integral equation technique to study water wave scattering by vertical permeable rigid barrier.

3. Mathematical formulation

The mathematical problem is studied in two-dimensional Cartesian coordinate system. Here, *y*-axis is taken in vertical direction and is positive downwards. Further, *x*-axis is considered in horizontal direction. Water occupies in the region $-\infty < x < \infty$ and $0 \le y < \infty$. The length of plate is $2d$. Assuming that the flow is of potential kind and also the motion is simple harmonic in time with angular frequency ω, the velocity potential $\Phi(x, y, t) = \mathrm{Re}\{\phi(x, y) e^{-i\omega t}\}$ exists and $\phi(x, y)$ satisfies

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0, \tag{1}$$

subject to the boundary conditions

$$\frac{\partial \phi}{\partial y} + K'\phi = 0 \quad \text{(Free surface BC)}, \quad K' = \frac{\omega^2}{g} \tag{2}$$

$$\phi, \nabla\phi \to 0 \text{ as } y \to \infty \quad \text{(Bottom Boundary condition)}, \tag{3}$$

Further, the boundary condition on the impermeable plate is given by

$$\frac{\partial \phi_1}{\partial x} = \frac{\partial \phi_2}{\partial x} = 0 \tag{4}$$

The conditions at infinity

$$\phi(x, y) \to \begin{cases} \phi^0(x, y) + R\phi^0(-x, y) & as\ x \to -\infty \\ T\phi^0(x, y) & as\ x \to \infty \end{cases} \tag{5}$$

where $\phi^0(x, y) = e^{-K'y + iK'x}$ is the incident wave velocity potential, and R and T are the unknowns associated with the reflection and transmission coefficients.

4. Method of solution

To formulate the integral equation, we have chosen the appropriate fundamental solution due to line source situated at (ξ, η) to apply the Green's theorem

$$G(x, y; \xi, \eta) = \frac{1}{2\pi} \ln \frac{(x-\xi)^2 + (y-\eta)^2}{(x-\xi)^2 + (y+\eta)^2} - \frac{1}{\pi} \int_{\Gamma} \frac{e^{-k(y+\eta)}}{k - K'} \cos k(x-\xi)\, dk \tag{6}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ is the region enclosing the plate and is given by

$\Gamma_1 : y = 0, -x' \le x \le x'; \Gamma_2 : x = x', 0 \le y \le y'; \Gamma_3 : y = y', -x' \le x \le x'; \Gamma_4 : x = -x', 0 \le y \le y'$, and consider the small circle radius ε and centered at (ξ, η). Here, for both x', y' tends to ∞ then the integral representation in (6) is given by (see Parsons and Martin (1992))

$$\int_c \frac{e^{-k(y+\eta)}}{k - K'} \cos k(x-\xi)\, dk = -e^{-K'(y+\eta)}\{(\ln K'\sqrt{(x-\xi)^2 + (y+\eta)^2} - \pi i + 0.5772) \times \cos K'(x-\xi) +$$

$$\tan^{-1}\left(\frac{x-\xi}{y+\eta}\right) \sin K'(x-\xi)\} + \sum_{m=1}^{\infty} \sum_{n=1}^{m} \frac{\left(-K'\sqrt{(x-\xi)^2 + (y+\eta)^2}\right)^m}{m!} \times \left(\frac{1}{n}\right) \cos m \tan^{-1}\left(\frac{x-\xi}{y+\eta}\right)$$

Assuming the scattered potential as $\phi^s(x, y) = \phi(x, y) - \phi^0(x, y)$. Now, apply the Green's integral theorem to scattered potential $\phi^s(x, y)$ and $G(x, y; \xi, \eta)$ which yields

$$\phi^s(\xi, \eta) = -\int_{\Gamma} [\phi^s](y) \frac{\partial G}{\partial x}(\xi, \eta; 0, y)\, dy. \tag{7}$$

However $[\phi^s](y) = [\phi](y)$ when y belongs to the gap. The BVP is converted into a hypersingular integral equation with jump discontinuity across the plate which is given by (Martin and Rizzo (1989) and Parsons and Martin (1992))

$$\phi(\xi, \eta) = \phi^0(\xi, \eta) - \int_{\Gamma} [\phi](y) \frac{\partial G}{\partial x}(\xi, \eta; 0, y)\, dy \tag{8}$$

It is observed that the unknown potential function gives the value zero at the ends of the plate to obtain the hypersingular integral equation. The same can be obtained by differentiating (8) with respect to ξ

$$\int_{\gamma} [\phi](y) \frac{\partial G}{\partial \xi \partial x}(0, \eta; 0, y)\, d\eta = iK' e^{-K'\eta} \tag{9}$$

where γ is barrier and we obtain

$$\frac{\partial^2 G}{\partial x \partial \xi} = -\frac{1}{(s-t)^2} + \kappa(s,t), \quad \kappa(s,t) = d^2 \left[\frac{y'^2 - x'^2}{(y'^2 + x'^2)^2} - \frac{2K'y'}{(y'^2 + x'^2)} - 2K'^2 \int_{\Gamma} \frac{e^{-k(y+\eta)}}{k - K'} \cos k(x-\xi)\, dk \right] \tag{10}$$

in which $x' = x - \xi, y' = y + \eta$. Substitute (9) into (8), we obtain integral equations where unknown discontinuous potential function as $f(t)$

$$\int_{-1}^{1} \frac{f(t)}{(s-t)^2} dt + \int_{-1}^{1} f(t)\kappa(s,t)dt = h(s), -1 < s < 1 \tag{11}$$

in the above equation first integral is considered as Hadamard finite part integral**.** In order to solve this hypersingular integral equation we adopted the collocation method, such that $f(\pm 1) = 0$ and the unknown function is approximated as

$$f(t) = \sqrt{(1-t^2)} \sum_{i=0}^{N} a_i U_i(t), \ (i = 0,1,..,N) \tag{12}$$

where $U_i(t)$Chebyshev polynomials of second kind, a_i are to be determined. Finally, we obtain the system of equations of the form

$$\sum_{n=0}^{N} a_n A_n(\mathrm{s}_j) = \mathrm{h}(\mathrm{s}_j), \ (\mathrm{j} = 0,1,\ldots,N), \ \mathrm{s}_j = \cos \frac{2j+1}{2(N+1)} \tag{13}$$

Where $A_n(\mathrm{s}) = -\pi(n+1)U_n(\mathrm{s}) + \int_{-1}^{1} \sqrt{(1-t^2)} U_n(\mathrm{t})\kappa(s_j,t)dt, \quad h(s) = iK'\mathrm{d}\, e^{-K'\eta} \ (n = 0,1,\ldots,N)$

The above integration cannot be evaluated analytically and so we adopt some numerical methods to evaluate the integration. Once we will be able the find unknown coefficients, we can find Reflection and Transmission coefficients.

5. References

[1] Chakraborty, R., & Mandal, B. N. (2014). Scattering of water waves by a submerged thin vertical elastic plate. Archive of Applied Mechanics, 84(2), 207-217.

[2] Evans, D. V. (1970). Diffraction of water waves by a submerged vertical plate. Journal of Fluid Mechanics, 40(3), 433-451.

[3] Gayen, R., &Mondal, A. (2014). A hypersingular integral equation approach to the porous plate problem. Applied Ocean Research, 46, 70-78.

[4] Kaya, A. C., & Erdogan, F. (1987). On the solution of integral equations with strongly singular kernels. Quarterly of Applied Mathematics, 45(1), 105-122.

[5] Krishnasamy, G., Schmerr, L. W., Rudolphi, T. J., & Rizzo, F. J. (1990). Hypersingular boundary integral equations: some applications in acoustic and elastic wave scattering. Journal of Applied Mechanics, 57(2), 404-414.

[6] Kuznetsov, N., Porter, R., Evans, D. V., & Simon, M. J. (1998). Uniqueness and trapped modes for surface-piercing cylinders in oblique waves. Journal of Fluid Mechanics, 365, 351-368.

[7] Martin, P. A., & Rizzo, F. J. (1989). On boundary integral equations for crack problems. Proceedings of the Royal Society of London. A. Mathematical and Physical Sciences, 421(1861), 341-355.

[8] Mandal, B. N., &Gayen, R. (2002). Water-wave scattering by two symmetric circular-arc-shaped thin plates. Journal of engineering mathematics, 44(3), 297-309.

[9] Parsons, N. F., & Martin, P. A. (1992). Scattering of water waves by submerged plates using hypersingular integral equations. Applied Ocean Research, 14(5), 313-321.

[10] Chakraborty, R., Mondal, A., & Gayen, R. (2016). Interaction of surface water waves with a vertical elastic plate: a hypersingular integral equation approach.

[11] Gupta, S., & Gayen, R. (2019). Water wave interaction with dual asymmetric non-uniform permeable plates using integral equations. Applied Mathematics and Computation, 346, 436-451.