A study of stress profiles in cyclic bending of an elasto-plastic beam

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ABSTRACT

In complex bending applications, quite often elasto-plastic deformation is experienced in structures. To model this elasto-plastic deformation, elastic perfectly-plastic stress-strain model is generally considered. In the case of elastic bending problems, it is well known that moment –curvature based constitutive law is over the stress-strain law, for ease of solving the governing deflection equations. However, in the case of inelastic bending, a nontrivial transformation occurs between stress-strain model and moment-curvature based model. In the process of obtaining the moment-curvature law, a detailed through thickness stress profile is necessary for any given curvature. In this paper, a study is presented on stress profiles along the depth of the beam at different stages of cyclic loading subjected to elasto-plastic deformation. The results indicate interesting stress profiles which are observed at points of zero moment and curvature in an elasto-plastic loading cycle.

Keywords: Moment-curvature, stress-profile, beam bending, elastic-perfectly plastic

1. INTRODUCTION

The primary purpose of beams as structural components is to support bending. The beam bending issue that is of interest for research is almost entirely non-linear in nature and results from material and/or geometric non-linearity. Researchers have devised various analytical, semi-analytical and numerical methods to solve such problems. Most analytical and semi-analytical approaches established in the last few decades to address non-linear bending problems are based on techniques that were originally developed from elliptic integrals, series expansion, homotopy perturbation etc. and can be found in [1–6] and the references therein. The finite element method (FEM) is possibly the most efficient method when it comes to purely numerical techniques. In this regard works mentioned in [7-11] is worth consulting. However, efforts to develop a non-FEM alternative that is more computationally affordable, effective, quick, and focused on beam mechanics have continued. Few methods are developed to solve elastic problems are mentioned in [12-19] Notable methods for resolving non-linear material problems can be found in [16,18] and the references therein.

Bending is primarily associated with moment and curvature; many formulations make explicit use of this relationship. Most often, using this approach, problems related to material nonlinearity are addressed. This renders the moment-curvature approach's applicability quite clear. Since in most of the times, a material is experimentally tested for its uniaxial stress-strain response, for an elasto-plastic case, the transformation into a moment-curvature based model is non-unique and non-trivial in such cases, most often one converts the inelastic uniaxial stress-strain result or model into an equivalent moment-curvature relationship, depending on the cross-sectional geometry of the beam. A method to obtain moment-curvature relationship from uniaxial stress strain law of standard material models can be found in [20]. Results are obtained for rectangular and circular beam cross-section. One of the main conclusions of this mentioned work is that the moment-curvature response obtained from the elastic-perfectly plastic material model displayed near kinematic hardening behavior. Although numerous studies on elasto-plastic bending have been reported, as can be seen in the papers mentioned above, only few studies are available which are focused on determination of through thickness stresses in beams. Especially, in cyclic loading, at any unloaded state, due to permanent deformation, complex stress profile exists within the beam. The approach is constrained to thin inextensible beams that conform to the Euler-Bernoulli hypothesis under small strain conditions, but it could be expanded to cases including finite rotation (curvature).

2. METHODOLOGY

In this section the method of obtaining the stress profile across the depth of a beam at various stages of cyclic bending is presented briefly (the detailed account can be found in [20]). In this method, the beam is assumed to be divided into an even number of imaginary layers in the depth direction, with the assumption that the uniaxial stress-strain law is valid only at the centroid of the layers. As a result, every centroid line will experience an increase in strain due to an increase in curvature. This will result in an increase in stress as computed from the constitutive law. This increment in stress will result in the increment of moment about the centroid of the beam cross-section. The total moment increment caused by the curvature increase for the beam section will then be calculated by adding all such moment increments numerically for each layer. This step results in the generation of moment-curvature response. As the moment curvature is obtained by integrating the stress strain response across the beam depth, the stress profile across a given section at any stage of loading can be easily accessed.

To illustrate the moment-curvature response a rectangular and a circular beam cross-section of elasto-plastic material model are considered. Rectangular beam cross-section with width b and depth h and a circular cross-section of diameter h are considered as shown in fig 2 and 3. The beams are subjected to pure bending under curvature-controlled loading as shown in Fig 1. A sinusoidal curvature input is applied after normalizing it with respect to the yield curvature κ_0. $\kappa_0 = 2\sigma_0/Eh$. Where σ_0 =yield stress of the material and E=young's modulus, and h is either the depth or diameter of the cross-section as the case may be. Each of the beam cross-section is discretized into even number of imaginary layers of uniform thickness along the depth direction. The vertical coordinate axis designated by Y, originate from the centroid of the entire cross-section. Ensuring plane section to remains plane the increment in strain is

$$\Delta\varepsilon = \Delta\kappa \times Y \tag{1}$$

Where $\Delta\varepsilon$ = increment in total strain at a distance y from neutral axis, $\Delta\kappa$=increment in curvature.

The increment in strain is then used to obtain the increment in stress by using an appropriate material model. In this study elastic perfectly plastic material model is considered as illustrated in Fig 4.

The incremental stress-strain relationship is given by:

$$\Delta\sigma = E(\Delta\varepsilon - \Delta\varepsilon^p) \tag{1}$$

Where $\Delta\sigma =$ increment in stress, $\Delta\varepsilon =$ increment in total strain, $\Delta\varepsilon^p =$ increment in plastic strain.

To obtain $\Delta\varepsilon^p$, a yield function f is defined as:

$$f = |\sigma| - \sigma_0 \tag{3}$$

Where σ_0 = yield stress.

And plastic load step check function g is defined as:

$$g = \sigma\Delta\varepsilon \tag{4}$$

The condition applied to the model to obtain plastic strain increment reads:

$$\text{If } f = 0 \text{ and } g > 0, \text{ then } \Delta\varepsilon^p = \Delta\varepsilon, \text{ else } \Delta\varepsilon^p = 0 \tag{5}$$

Subsequently, after determination of the stress increment, the increment in moment for the rectangular section reads:

$$\Delta M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta \sigma Y b dY \tag{6}$$

And for circular section it is defined as:

$$\Delta M = \int_{-\frac{h}{2}}^{\frac{h}{2}} \Delta \sigma Y \sqrt{\left(\frac{h}{2}\right)^2 - Y^2}\, dY \tag{7}$$

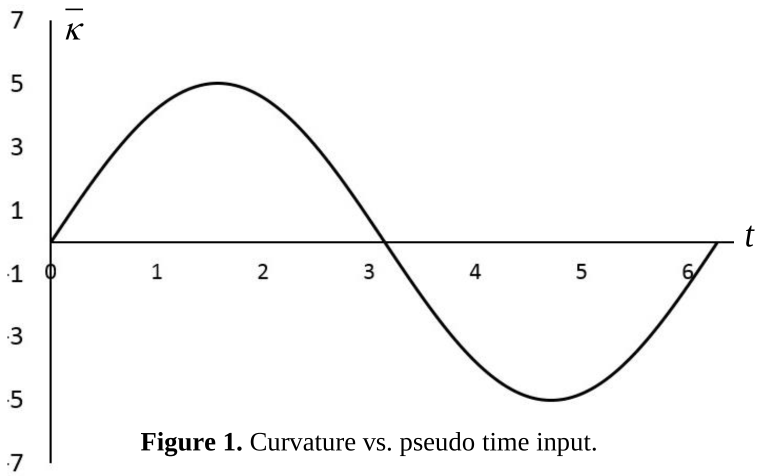


Figure 1. Curvature vs. pseudo time input.

$$\bar{\kappa} = \frac{\kappa}{\kappa_0}, \kappa_0 = \frac{2\sigma_0}{Eh}$$

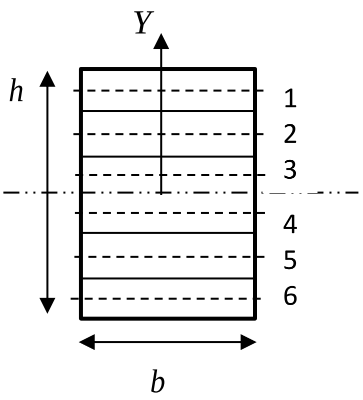


Figure 2. Layering scheme of rectangular beam cross-section (Dotted line indicates centroidal axis of particular layer and solid line are the layer's boundary)

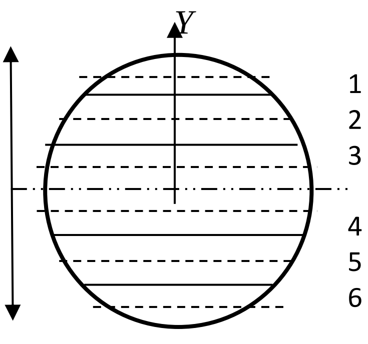


Figure 3. Layering scheme of circular beam cross-section (Dotted line indicates centroidal axis of particular layer and solid line are the layer's boundary)

3. RESULTS

In accordance with the method discussed in the previous section, the stress at each layer corresponding to any curvature level is known. In Fig 5, the normalized moment $\overline{M}$ (with respect to yield moment $M_0 = \sigma_0 bh^2 / 6$ for rectangular cross-section, $M_0 = \sigma_0 \pi h^3 / 32$ for circular cross-section) is plotted against the normalized curvature $\overline{\kappa}$ (with respect to yield curvature $\kappa_0 = 2\sigma_0 / Eh$) for a rectangular beam cross-section, discretized into 100 number of layers. Clearly, the well-known result of the asymptote value of 1.5 is obtained. Also, it can be seen that the response is kinematic in the moment curvature space. A very similar response with a different asymptote (value of 1.7) is obtained for a circular cross-section as shown in Fig 4. Also, as previously explained (in section 2), for any point on the curve of Fig.5 and Fig.6 the stress profile across the cross-section may be obtained easily. From Fig. 5 and 6, conjugate points are identified for same magnitude of curvature and named as A-D, B-E and C-F. The corresponding stress profiles are subsequently obtained and are given in Fig. 7, 8 and 9 for the rectangular cross-section and Fig. 10, 11 and 12 for the circular cross-section. Clearly, at A and at D, the beam is in positive and negative plastic loading states, respectively (sign chosen according to the sign of curvature). The corresponding stress profiles are shown in Fig.7 and 10. The points, B and E are points which have zero moment but have been previously plastically deformed. They correspond to the complex residual stress profiles as shown in Fig. 8 and 11 respectively for rectangular and circular sections respectively. The points C and F correspond to zero curvature states which have been previously plastically deformed as shown in Fig. 9 and 12. It can be seen that the conjugate points correspond to stress profile which are mirror image about the depth axis. The number of points of zero stress or the size of zero stress region on the depth axis appears to increase as elasto-plastic loading progresses. It is interesting to note here that there is a considerable region around the neutral axis which is stress free. It is also observed from the stress profile that non-dimensional stress at top and bottom most layer of circular beam cross-section (x2 in Fig 11= 0.64025) is 40% more compared to rectangular cross-section (x1 in Fig 8= 0.46007) at zero curvature state.

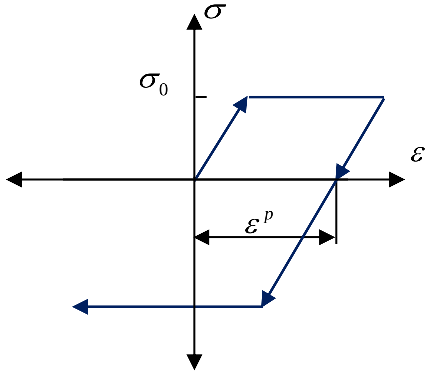


Figure 4. Stress-Strain diagram for elastic-perfectly plastic material model

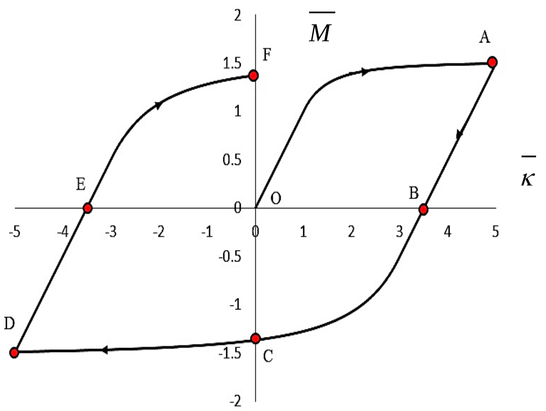


Figure 5. Moment-Curvature relation from perfectly plastic material model for rectangular cross-section

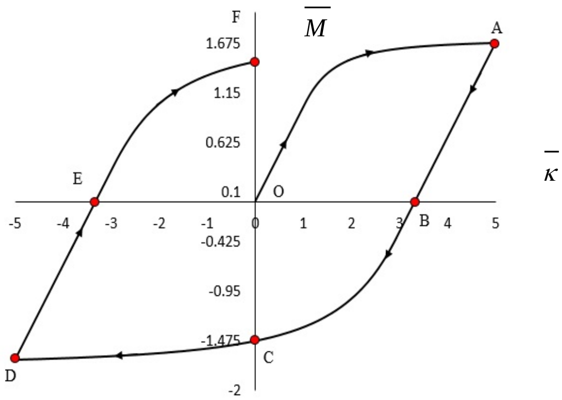


Figure 6. Moment-Curvature relation from perfectly plastic material model for rectangular cross-section

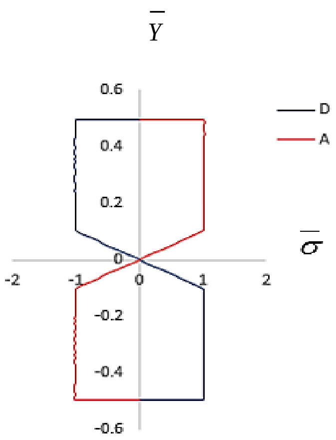


Figure 7 Stress profile at point A D for rectangular cross-section

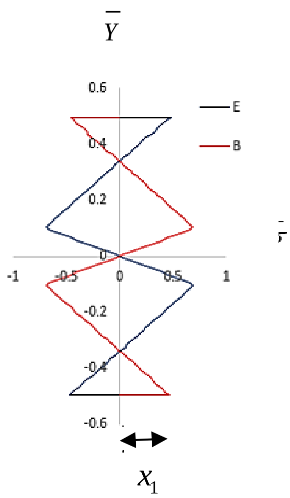


Figure 8 Stress profile at point E B for rectangular cross-section

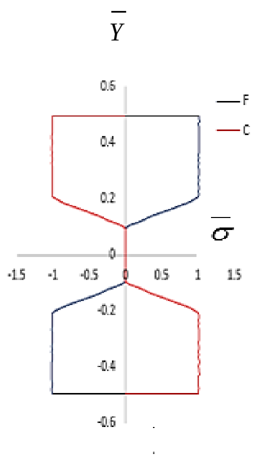
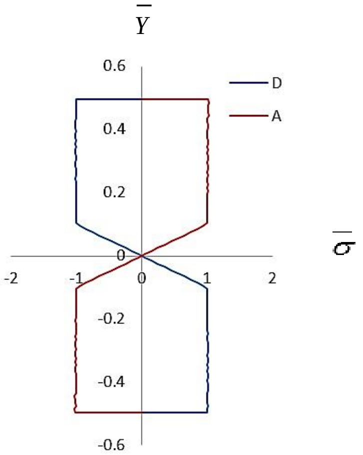
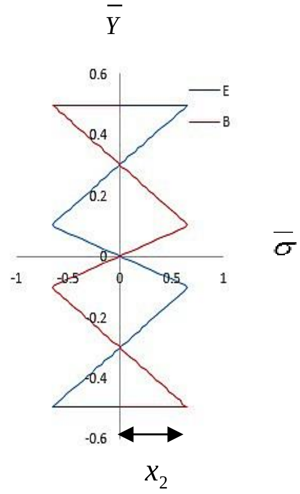


Figure 9 Stress profile at point F C for rectangular cross-section





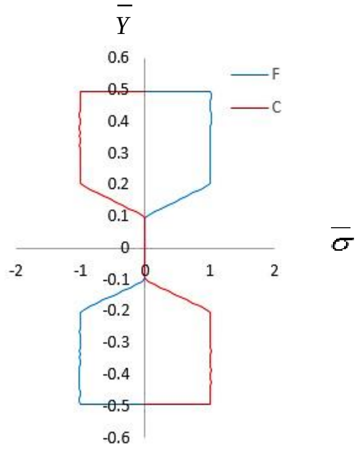


Figure 10 Stress profile at point A D for circular cross-section

Figure 11 Stress profile at point E B for circular cross-section

Figure 12 Stress profile at point F C for circular cross-section

5. CONCLUSION

Since the moment curvature response is obtained by integrating the stress-strain response from the discretized beam cross-section, the corresponding stress profile across beam depth are easily accessed. Additionally, the profiles obtained are more accurate than approaches where moment curvature responses are considered in ad.hoc manners; like elastic perfectly plastic or elastic linearly hardening moment-curvature models. The stress profiles obtained show that, for the conjugate points on the moment-curvature plot, the stress profiles are mirror images of each other. Further, for the zero curvature conjugate points, it is found that there is a considerable region around the neutral axis which is free from stresses for both rectangular and circular section. This phenomenon is akin to a prestress condition. And since, portion of the beam is free from stresses; it can be further loaded to take more load. Also, it is found that stress at extreme top and bottom layer of circular section (x_2 in Fig 11) is more compared to the rectangular section (x_1 in Fig 8) for zero moment state in moment –curvature response. In future, other elasto-plastic material models and different cross-sections will be considered in the study.

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