# Fractional Cattaneo-type telegraph equation in axisymmetric and central symmetric with time-harmonic source and associated thermal stresses

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## 1. ABSTRACT

Time-nonlocal generalization of the classical Fourier law with the Short-tale memory an exponential kernel can be interpreted in terms of fractional calculus and leads to the time-fractional Cattaneo equation. Cattaneo-type telegraph equation and time-fractional Cattaneo-type telegraph equation with the harmonic source term under zero initial conditions is studied in axisymmetric case and central symmetric case. The fundamental solutions are obtained by using the integral transform techniques. The corresponding thermal stresses are found using the displacement potential. Numerical results are illustrated graphically for different values of nondimensional parameters.

## 2. RESEARCH AIM

1. To study Integer order Cattaneo-type telegraph equation with a harmonic source term under zero initial conditions in axisymmetric case and central symmetric case.

2. To study time-fractional Cattaneo-type telegraph equation with a harmonic source term under zero initial conditions in axisymmetric case and central symmetric case.

3. To determine thermal stresses by using displacement potential function.

## 3. BRIEF LITERATURE SURVEY

Angstrom [1] was the first to examine the heat conduction equation under the time-harmonic impact. Povstenko [2] examined fractional heat conduction in a space with a source varying harmonically in time, and the corresponding thermal stresses. The telegraph equation has two harmonic wave solutions, which are shown in [3] i.e temporally attenuated and spatially periodic (TASP), and spatially attenuated and spatially periodic (SATP).

## 4. PROBLEM FORMULATION

Short-tale memory with an exponential kernel

$$\boldsymbol{q}(t) = -\frac{\kappa}{\tau_0}\int_0^t \exp\left(-\frac{t-\tau}{\tau_0}\right) \mathrm{grad}\, T(\tau)\, d\tau \tag{1}$$

where $\tau_0$ is a non-negative constant, and $\kappa$ is the thermal conductivity of a solid, or in the Cattaneo form [4]

$$\boldsymbol{q} + \tau_0 \frac{\partial \boldsymbol{q}}{\partial \mathrm{t}} = -\kappa\, \mathrm{grad}\, T \tag{2}$$

leads to the telegraph equation for temperature

$$\frac{\partial \mathrm{T}}{\partial \mathrm{t}} + \tau_0 \frac{\partial^2 \mathrm{T}}{\partial \mathrm{t}^2} = \mathrm{a}\, \Delta \mathrm{T} \tag{3}$$

where $a$ is the thermal diffusivity.

Consider the following generalization of equation (2) in the form

$$I^{1-\alpha}\mathbf{q} + \tau_0 \frac{\partial^\alpha \mathbf{q}}{\partial t^\alpha} = -\kappa \,\mathrm{grad}\, T \qquad 0 < \alpha \leq 1 \qquad (4)$$

leads to the generalized Cattaneo equation III

$$\frac{\partial^\alpha T}{\partial t^\alpha} + \tau_0 \frac{\partial^{1+\alpha} T}{\partial t^{1+\alpha}} = a\,\Delta T \qquad 0 < \alpha \leq 1 \qquad (5)$$

where $\frac{\partial^\alpha T}{\partial t^\alpha}$ is the Caputo [5] fractional derivative of order $\alpha$.

## 4. SOLUTION METHODOLOGY

The solution of Integer order Cattaneo-type telegraph equation (3) and time-fractional Cattaneo-type telegraph equation (5) with a harmonic source term can be obtained under zero initial conditions in axisymmetric case and central symmetric case by using integral transforms techniques. The displacement potential is used to determine the corresponding thermal stresses. Numerical results are illustrated graphically for various values of nondimensional parameters.

## 5. CONCLUSIONS

1. Integer order Cattaneo-type telegraph equation and time-fractional Cattaneo-type telegraph equation with a harmonic source term is examined under zero initial conditions in axisymmetric case and central symmetric case. The solutions are obtained by using the Laplace, Hankel and Fourier transform techniques. The corresponding thermal stresses have also been examined.

2. To show the differences between the fractional model and the classical integer model, the solutions for various values of fractional order $\alpha$ are shown in figures. The numerical results demonstrate the significant effect on the temperature distribution along angular frequency and radial coordinate, and also stress distribution along radial coordinate.

3. If the source term can be expanded into a Fourier series then the acquired solutions can be successfully utilised. The solutions can be represented as a superposition of harmonic terms.

4. The obtained solutions can be used in the Medical science such as radioactive therapy, laser technology, flash burns of human skin etc.

## REFERENCES

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