MQ based RBF-HFD method for solving unsteady convection-diffusion equation

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Abstract

A fourth order Multiquadric (MQ) based RBF-HFD scheme is used to discretize the spatial derivatives in unsteady 1D convection–diffusion equations. Then the semi-discredited equations are solved using fourth order Runge-Kutta method. By minimizing the maximum of the local truncation errors, we obtain a constant optimal value of the shape parameter appearing in the MQ radial function.

1 RESEARCH AIM & PROBLEM FORMULATION

In this work, we aim to numerically solve 1-D convection-diffusion equation problem from [5] using fourth order RBF-HFD approximations and RK-4 time stepping schemes.

$$u_t + cu_x = \nu u_{xx},\ 0 \leq x \leq 1,\ 0 \leq t \leq T \tag{1}$$

where c and ν are positive constants. The exact solution for (1) is

$$u(x,t) = \frac{1}{\sqrt{1+4t}} exp\left(-\frac{(x-0.2-tc)^2}{\nu(1+4t)}\right). \tag{2}$$

2 BRIEF LITERATURE SURVEY

Convection-diffusion equations are broadly used for modelling and simulations of various complex phenomena in science and engineering (Hundsdorfer & Verwer; Morton, [3]). Since for most application problems, it is impossible to solve convection-diffusion equations analytically, efficient numerical algorithms are becoming increasingly crucial to numerical simulations involving convection-diffusion equations. Recently a great deal of effort has been devoted to developing high-order compact schemes, which utilise only the grid nodes directly adjacent to the central node. Noye and Tan [1] derived a class of high-order implicit schemes for solving the one-dimensional unsteady convection-diffusion equations. This method is very stable and accurate (third-order in space and second-order in time). Gupta *et al.* [2]) proposed a fourth-order finite difference scheme for steady convection-diffusion equation with variable coefficients.

3 SOLUTION METHODOLOGY

MQ based fourth-order RBF-HFD formulas for first and second derivatives ([4]) are used at interior and boundary nodes for semi-discretization. Boundary nodes formulas :

$$u'(x_1) \approx \alpha_{11}\, u_1 + \alpha_{12}\, u_2 + \alpha_{13}\, u_3 + \alpha_{14}\, u_4 + \beta_{11}\, u_2', \tag{3}$$

$$u''(x_1) \approx \alpha_{21}\, u_1 + \alpha_{22}\, u_2 + \alpha_{23}\, u_3 + \alpha_{24}\, u_4 + \beta_{21}\, u_2''. \tag{4}$$

Then Runge-Kutta 4 method is used for time discretization. The maximum error for Example (1) are reported in Table 1.

Table 1: Maximum errors for Example (1): $c = 1$, $\nu = 1$

No. of Nodes	Different Final time (T)				
		$T = 0.25$	$T = 0.5$	$T = 0.75$	$T = 1.0$
$N = 21$	Compact FD (order 4)	1.3807e-07	4.3811e-08	9.5732e-09	4.7734e-09
	ϵ_e (exact)	0.3865	0.3121	0.2705	0.2233
	RBF-HFD (order 4)	2.2857e-08	6.8780e-09	2.0928e-09	5.4354e-10
	ϵ (approx.)	0.3565	0.2997	0.2565	0.2285
	RBF-HFD (order 4)	2.6484e-08	7.2224e-09	2.5696e-09	7.6612e-10
	$\lvert\epsilon - \epsilon_e\rvert$	0.0300	0.0124	0.0140	0.0052

4 SIGNIFICANT CONCLUSIONS

We have developed a fourth order RBF-HFD method for the numerical solution of unsteady 1D convection–diffusion problems. The method is fourth-order accurate in both time and space. We also obtain a constant optimal value of the shape parameter (ϵ) for which accuracy is further improved.

References

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