Gaussian based RBF-HFD method for time-fractional convection-diffusion equation

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Abstract

High order numerical scheme is developed for solving time fractional convection-diffusion equation. Gaussian (GA) based fourth order RBF-HFD formulas are used for spatial discretization of convection and diffusion terms. Then high order time discretization scheme is used to approximate the fractional order Caputo derivative. We obtain better accuracy as compared compact FD and RBF-FD schemes.

1 RESEARCH AIM & PROBLEM FORMULATION

In this work, we numerically solve time fractional convection-diffusion problem of the form

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u_{xx}(x,t) - u_x(x,t) + f(x,t), \quad (x,t) \in (0,1) \times [0,T] \tag{1}$$

where $0 < \alpha < 1$, f and u are smooth functions, the initial and boundary conditions respectively are $u(x,0) = u_0(x)$, and $u(0,t) = g_1(t)$, $u(1,t) = g_2(t)$. The fractional derivative $\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}$ is Caputo fractional derivative of order α.

2 BRIEF LITERATURE SURVEY

In recent years there has been a growing interest in the field of fractional calculus for the convection-diffusion model because this phenomena is found in many natural or physical situations such as mass, heat and other transport processes. Exact closed-form solutions of fractional partial differential equations (FPDEs) are difficult to obtain using analytical methods, and it is even more difficult to solve high-dimensional fractional order PDE problems numerically. The increasing use of computational fluid dynamics (CFD) for engineering design and analysis demands highly efficient numerical methods. Efficiency is of **paramount** importance for successful implementation of the numerical methods. Several authors have proposed a variety of numerical methods for the model problem, such as finite element methods, finite difference methods, spectral methods and meshless methods. A new fractional numerical differentiation formula (L1-2 formula) of order $(3-\alpha)$ developed by Guang-hua Gao *et.al.* [3]. Later higher order approximations to Caputo derivative of order $(4-\alpha)$ is established by Jianxiong Cao *et.al.* [2]. The higher order schemes are applied to solve the Caputo type advection-diffusion equation with Dirichlet boundary conditions in [4].

3 SOLUTION METHODOLOGY

Consider a partition of the interval $[0, T]$ into M sub-intervals $[t_{j-1}, t_j]$, $j = 1, 2, \cdots M$ with equi-spaced nodes $t_j = j\Delta t, j = 0, 1, \cdots, M$. Here $\Delta t = T/M$. Then to discretize the Caputo time-fractional derivative for solving (1), we used the following approximation

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{(\Delta t)^{1-\alpha}}{\Gamma(2-\alpha)} \sum_{j=1}^{k} a_{k-j}^{(\alpha)} \delta_t f_{j-\frac{1}{2}} + \frac{(\Delta t)^{2-\alpha}}{\Gamma(2-\alpha)} \sum_{j=2}^{k} b_{k-j}^{(\alpha)} \delta_t^2 f_{j-1} + \frac{(\Delta t)^{3-\alpha}}{\Gamma(4-\alpha)} \sum_{j=3}^{k} c_{k-j}^{(\alpha)} \delta_t^3 f_{j-\frac{3}{2}},$$

where $\delta_t f_{j-\frac{1}{2}}, \delta_t^2 f_{j-1}$ and $\delta_t^3 f_{j-\frac{3}{2}}$ are the difference operators defined as

$\delta_t f_{j-\frac{1}{2}} = \frac{f(t_j)-f(t_{j-1})}{\Delta t}, \delta_t^2 f_{j-1} = \frac{f(t_j)-2f(t_{j-1})+f(t_{j-2})}{(\Delta t)^2}$ and $\delta_t^3 f_{j-\frac{3}{2}} = \frac{f(t_j)-3f(t_{j-1})+3f(t_{j-2})+f(t_{j-3})}{(\Delta t)^3}$.

For the spatial derivative discretization we have used the fourth order GA based RBF-HFD formulas [1], where the approximation of first and second derivative are defined as follows. Then fourth order formula to approximate first and second derivative at x_0 are defined as follows.

$$u'(x_0) \approx \alpha_{-1}u_{-1} + \alpha_0 u_0 + \alpha_1 u_1 + \beta_{-1}u'_{-1} + \beta_1 u'_1 \tag{2}$$

$$u''(x_0) = \alpha_{-1}u_{-1} + \alpha_0 u_0 + \alpha_1 u_1 + \beta_{-1}u''_{-1} + \beta_1 u''_1 \tag{3}$$

4 SIGNIFICANT CONCLUSIONS

- In this work, we use fourth order GA based RBF-HFD formulas to approximate spatial derivatives and high order $(4-\alpha)$ time stepping scheme for approximating fractional time derivative.
- Through some examples we show better accuracy with RBF-HFD method as compared with compact FD and RBF-FD schemes.

References

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