Operational Matrix Method for Fractional Differential Equations

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INTRODUCTION & OBJECTIVE

This paper aims to develop a spectral approach for solving time fractional differential equations using the derivatives of orthogonal polynomials. The technique involves utilizing operational matrices, which are matrices that encode the operations performed on the polynomials. This methodology offers a structured means to manipulate these polynomials, enabling their application in diverse mathematical and engineering contexts. Through the application of operational matrices, the original time fractional differential equation is transformed into a set of linear or nonlinear algebraic equations. The estimation of error bounds of numerical solutions is also studied. The numerical experiments have been performed over a few test examples to validate the proposed numerical method.
Recently most numerical techniques for fractional differential equations were built on a spectral method based on distinct basis functions like Legendre polynomials, Bernstein polynomials, etc. Motivated by such methods, this paper endeavours to solve various linear and nonlinear fractional differential equations of the following forms by considering a class of orthogonal polynomials under one umbrella.

$$f\left(\frac{\partial^{\alpha}\mathrm{u}}{\partial \mathrm{t}^{\alpha}}, \frac{\partial u}{\partial t}, \frac{\partial^{2}\mathrm{u}}{\partial x^{2}}, \frac{\partial u}{\partial x}, u, g\right) = 0$$

With the following initial and boundary conditions;
Initial condition : u(x,0)=h(x), $0 \leq x \leq I$,
Boundary conditions : u(0,t)=$h_1(t)$, u(I,t)=$h_2(t)$, $0 \leq t < T$,
Where α is the order of the derivatives and g is the source term.

RESULTS & HIGHLIGHTS OF IMPOINTANT POINTS

Example : Consider the following fractional differential equations;

$$\frac{\partial u^{\alpha}}{\partial t^{\alpha}} - \frac{\partial u^{2}}{\partial x^{2}} - u = g(x,t)$$

with source term g(x,t)=$2\frac{t^{2-\alpha}}{\Gamma(3-\alpha)}x(2-x) + x(2-x)t^2 + 2t^2$,

u(x,0)=0, $0 \leq x \leq 2$, u(0,t)=u(2,t)=0, $0 < t \leq 1$.

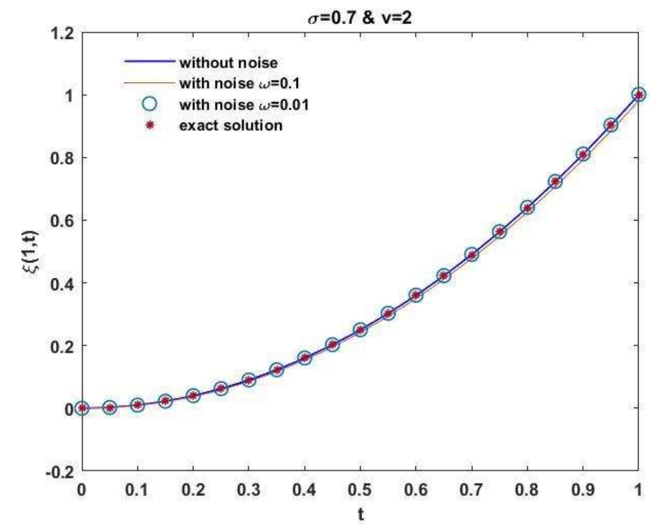
The source term is chosen so that for all α, u(x,t)=$t^2x(2-x)$ is the exact solution.

The comparative study is made with the existing method, and numerical stability analysis is performed by method of perturbation (Given in the following table and figure).

Table : Comparison of proposed method and method in [5] for α=0.7, t=0.4

x	Exact	Implicit Scheme	E-I scheme	I-E scheme	Presented Approach
0.4	0.102400	0.102406	0.102222	0.102222	0.102400
0.8	0.153600	0.153609	0.153318	0.153296	0.153600
1.2	0.153600	0.153609	0.153318	0.153296	0.153600
1.6	0.102400	0.102406	0.102222	0.102222	0.102400
CPU (s)	-	6.6	3.8	3.8	3.3

Figure : Comparison of solution with adding noises



Here an attempt is made to solve the equation via the operational matrix of Jacobi polynomials along with the collocation method. Specifically, the derivatives of Legendre polynomials, and Chebyshev polynomials are used to represent both integer and non integer derivatives of the function. If the resultant system is nonlinear, a Newton -like solver is applied, which is capable of handling nonlinear systems.

Our approach offers a user friendly experience with minimal computational time. The use of various special basis polynomials, derived from Jacobi polynomials, yields fairly comparable outcomes. Among these polynomials, utilizing Legendre polynomials leads to decreased computational time. Remarkable accurate results are attainable even for modest values of degree of polynomials. Through a comparative analysis, our study validates the superior accuracy of the proposed method when compared to existing approaches.

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