A Finite Volume Complete Flux Scheme for Time Fractional ADR Equation

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Introduction and Objectives

In recent years, there have been extensive investigations on fractional differential equations (FDEs), which have applications in physical, biological, geological, and financial systems. A series of numerical and theoretical study have been done by Yumin Lin et. al 2006[1], Mingrong Cui 2013[2], Toprakseven S. et. al 2021[3], Komal Taneja et. al 2023[4], and Yi Xu et. al 2023[5] for time fractional diffusion and advection-diffusion-reaction (ADR) equation recently. Here, we propose a novel finite-volume complete flux scheme for the following time fractional ADR equation.

$$ {}_0^C D_t^\gamma u(x,t) + \frac{\partial}{\partial x}\left(-a\frac{\partial u(x,t)}{\partial x} + bu(x,t)\right) + cu(x,t) = s(x,t) \quad l_1 \le x \le l_2, t \in [0,\mathrm{T}], \tag{1} $$

with initial condition and boundary conditions

$$ u(x,0) = w(x), l_1 \le x \le l_2, \text{and } \; u(l_1,t) = \phi_1(t), \;\; u(l_2,t) = \phi_2(t), t \in [0,\mathrm{T}] $$

where, a, b and c are the constant diffusion coefficient, advection velocity and the reaction coefficient respectively. The initial and boundary conditions will be stated while validating the scheme numerically. ${}_0^C D_t^\gamma(\cdot)$ is the Caputo fractional derivative given by $\gamma \in (0,1)$[2]

$$ {}_0^C D_t^\gamma u(x,t) = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{1}{(t-\tau)^\gamma} \frac{\partial u(x,\tau)}{\partial \tau}\, \mathrm{d}\tau. \tag{2} $$

We represent total flux by

$$ f(x,t) := -au_x(x,t) + bu(x,t) \tag{3} $$

and it is referred as the complete flux. The spatial discretization of the ADR equation, by finite volume method, requires the convective and the diffusion fluxes at the faces of the control volume. The semi-discretized finite volume

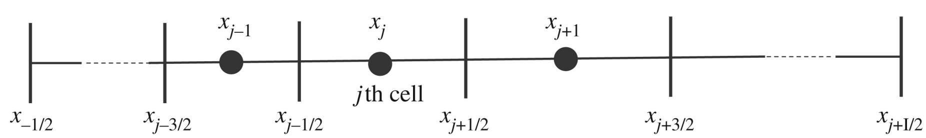

Figure 1: 1-D finite volume grid.

formulation can be given by

$$ {}_0^C D_t^\gamma u(x_j,t)\Delta x + f_{j+\frac{1}{2}}(t) - f_{j-\frac{1}{2}}(t) = \int_{x_{j-\frac{1}{2}}}^{x_{j+\frac{1}{2}}} (s(x,t) - cu(x,t))\, \mathrm{d}x, \tag{4} $$

where $\Delta x = x_{j+1} - x_j$ be the spatial grid spacing. We calculate the numerical flux $f_{j+\frac{1}{2}}$ by solving the following steady state local boundary value problem (BVP)

$$\frac{df}{dx} = s(x) - cu(x). \tag{5}$$

$$u(x_j) = u_j \,\text{and}\, u(x_{j+1}) = u_{j+1},$$

Where u_j is the finite volume numerical approximation of u at the j^{th} cell, x_j. The flux $f_{j+\frac{1}{2}}$ comes out to be a sum of homogeneous part and inhomogeneous part. Where the inhomogeneous part contains the influence of source terms. We get a fully discretize scheme by approximating the time fractional derivative in equation (4) by the following equation [6]

$$^C_0D^\gamma_t u(x_j,t_n) = \frac{\Delta t^{-\gamma}}{\Gamma(2-\Gamma)}\left[d_0 u^n_j - \sum_{l\leq n-1}(d_{n-l-1} - d_{n-1})u^l_j - d_{n-1}u^0_j\right] + o(\Delta t^{2-\gamma}), \tag{6}$$

where Δt is the time step size, $\gamma \in (0,1)$ and $d_j = (j+1)^{1-\gamma} - j^{1-\gamma}$.
We treat the source term and the complete flux term at explicit time for the time evolution.

Results and Important points

We propose a novel complete flux scheme based on finite volume method for time fractional ADR equation. The proposed numerical scheme will be first validated for $\gamma = 1$ against benchmark cases with exact solution. Also to check the ability of the scheme the influence of the important parameters like Peclet Number will be assessed. Next, we do numerical validation of the proposed scheme with the pre-existing know solutions for different time fractional order derivatives i.e. for $\gamma \in (0,1)$. The time evolution of the solution with different fractional order derivatives will be traced and the physics behind the obtained results will be analysed and discussed. The results will presented through x-y and surface plots.

References

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