A Combined Least-Squares and Finite Element Approximation for Optimal Transport problem using the Monge-Ampère Equation

Ram Manohar [1a], Anil Rathi [1a], J.H.M. Ten Thije Boonkkamp [1b] and B.V. Rathish Kumar [1a]

[a] Indian Institute of Technology Kanpur, Kanpur, India

[b] CASA, Eindhoven University of Technology, Eindhoven, The Netherlands

1. INTRODUCTION & OBJECTIVE

In this study, we present a numerical scheme based on least squares and finite element methods to compute the solution of the optimal transport problem with quadratic cost function in the 2D domain. The model problem is as follows: To find a map $\boldsymbol{m}: \boldsymbol{\mathcal{X}} \mapsto \boldsymbol{\mathcal{Y}}$ such that the minimization transportation cost functional is given by

$$\mathcal{C}[\boldsymbol{m}] = \iint_{\mathcal{X}} |\boldsymbol{x} - \boldsymbol{m}(\boldsymbol{x})|^2 f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \quad (1.1)$$

Additionally, m must satisfy, $\forall$ h (see, [3])

$$\iint_{\mathcal{X}} h(\boldsymbol{m}(\boldsymbol{x})) f(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} = \iint_{\mathcal{Y}} h(\boldsymbol{p}) g(\boldsymbol{p}) \mathrm{d}\boldsymbol{p}, \quad (1.2)$$

where $f: \boldsymbol{\mathcal{X}} \to [0, \infty)$ and $g: \boldsymbol{\mathcal{Y}} \to (0, \infty)$ are bounded functions, which denote the (mass) densities with bounded compact supports $\boldsymbol{\mathcal{X}} \subset \mathbb{R}^2$ and $\boldsymbol{\mathcal{Y}} \subset \mathbb{R}^2$ and $|\cdot|$ denotes the vector 2-norm. Practical applications of optimal mass transport are shape recognition in image processing [1], image registration, reflector design, mesh generation, astrophyics [2], etc.

According to a theorem by Brenier [1] says that such an optimal mapping m is the unique gradient of a convex function u, i. e., $\boldsymbol{m} = \nabla u$. Hence, the solution of the optimal transport problem with quadratic cost function as defined in (1.1) and (1.2), is the same as the solution of the Monge-Ampère equation

$$det(D^2 u) = \frac{f(x,y)}{g(\nabla u(x,y))}, \quad (x, y) \in \boldsymbol{\mathcal{X}}, \quad (1.3)$$

with boundary condition

$$\nabla u(\partial \boldsymbol{\mathcal{X}}) = \partial \boldsymbol{\mathcal{Y}}. \quad (1.4)$$

The proposed numerical scheme for optimal transport using the Monge-Ampère equation is an efficient and accurate numerical method for solving the optimal transport problem. It can handle non-convex target domains and has various benefits over current approaches. The method has been demonstrated to be effective through numerical experiments. It is one of the few numerical algorithms capable of solving this problem efficiently with the proper transport boundary condition. We provide an outline of the algorithm. The approach requires three minimization problems to be solved in each iteration. We begin with an initial guess of m^0. Subsequently, we perform the iteration

$$\boldsymbol{b}^{n+1} = \underset{b\in\mathcal{B}}{argmin}\, J_{\mathrm{B}}(\boldsymbol{m}^{n}, \boldsymbol{b}), \tag{1.5}$$

$$\boldsymbol{P}^{n+1} = \underset{\boldsymbol{P}\in\mathcal{P}(\boldsymbol{m}^{n})}{argmin}\, J_{I}(\boldsymbol{m}^{n}, \boldsymbol{P}), \tag{1.6}$$

$$\boldsymbol{m}^{n+1} = \underset{\boldsymbol{m}\in\mathcal{V}}{argmin}\, J(\boldsymbol{m}, \boldsymbol{P}^{n+1}, \boldsymbol{b}^{n+1}), \tag{1.7}$$

where the minimization boundary, interior and weighted functionals are defined as

$$J_{\mathrm{B}}(\boldsymbol{m}, \boldsymbol{b}) = \frac{1}{2}\oint_{\partial\boldsymbol{\mathcal{X}}} |\boldsymbol{m} - \boldsymbol{b}|^{2}\, \mathrm{d}s, \tag{1.8}$$

$$J_{I}(\boldsymbol{m}, \boldsymbol{P}) = \frac{1}{2}\iint_{\boldsymbol{\mathcal{X}}} \parallel \mathrm{D}m - \boldsymbol{P} \parallel^{2}\ \mathrm{d}x\, \mathrm{d}y, \tag{1.9}$$

norm ‖ . ‖ refers the Fr$\ddot{o}$benius norm, and

$$J(\boldsymbol{m}, \boldsymbol{P}, \boldsymbol{b}) = (1-\alpha)J_{\mathrm{B}}(\boldsymbol{m}, \boldsymbol{b}) + \alpha J_{\mathrm{I}}(\boldsymbol{m}, \boldsymbol{P}), \tag{1.10}$$

respectively, where $\alpha \in (0,1)$. We minimize these functionals over the sets

$$\mathcal{B} = \{\boldsymbol{b} \in [C^{1}(\partial\boldsymbol{\mathcal{X}})]^{2} | \boldsymbol{b}(\boldsymbol{x}) \in \partial\boldsymbol{\mathcal{Y}} \quad \forall\boldsymbol{x} \in \partial\boldsymbol{\mathcal{X}}\}, \tag{1.11}$$

$$\mathcal{P}(\boldsymbol{m}) = \left\{\boldsymbol{P} \in [C^{1}(\boldsymbol{\mathcal{X}})]^{2\times2}_{\mathrm{spsd}} | det(\boldsymbol{P}(x,y)) = \frac{f(x,y)}{g(\boldsymbol{m}(x,y))}\right\}, \tag{1.12}$$

and $$\mathcal{V} = [C^{2}(\boldsymbol{\mathcal{X}})]^{2}, \tag{1.13}$$

respectively. The minimizers b^{n+1}, P^{n+1}, and m^{n+1} are calculated by repeatedly using the minization cost functionals J_B, J_I, and J over the sets $\mathcal{B}$, $\mathcal{P}(\boldsymbol{m})$, and $\mathcal{V}$, respectively.

2. RESULTS

We now conclude and summarize our discussion of the technique and demonstrate how to determine the solution u of the Monge-Ampère equation from the ideal transportation mapping m. Our interest in optimal transport comes from the field of illumination optics, which concerns the design of lenses and reflectors for use in lighting. We shall demonstrate that a solution of (1.3) with boundary condition (1.4) accurately describes the geometry of a lens or reflector surface $z = u(x,y)$ dispersing the light from a parallel beam into a specified output light distribution. The parallel light beam is represented by the function $f(x,y)$, while the output distribution is represented by the function $g(\nabla u)$. We present several numerical experiments to demonstrate the effectiveness of the proposed method. We will compare our results to those of current methods, such as the viscosity solution method and the numerical solution of the optimal transportation problem based on the Monge-Ampère equation. The computation time scales well with the grid size and has the additional advantage that the target domain may be non-convex. The experiments show that the proposed method is faster than the existing methods and more accurate in solving the optimal transport problems.

REFERENCES

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