Impact of ocean water density stratification on the propagation of acoustic-gravity waves

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1. INTRODUCTION & OBJECTIVE

Acoustic-gravity waves (AGWs) are low-frequency compression waves that occur in the ocean, typically resulting from fluctuations in pressure fields caused by movements along the seabed. These waves, travelling at speeds close to that of sound, are nearly ten times faster than tsunamis or rogue waves, making them an essential element of early tsunami detection systems [1]. AGWs are introduced by the ocean's slight compressibility, which gives rise to their sinusoidal vertical profiles [2]. Although AGWs are extensively studied, accurately quantifying their existence and propagation within a stratified ocean presents ongoing challenges. Conventional ocean hydrodynamics often assume uniform water density, yet variations in salinity and temperature can lead to the formation of a two-layer density structure in certain conditions [3]. This research investigates the impact of ocean stratification on AGW propagation by addressing a boundary value problem through linearised water wave theory. By analysing the resulting fast (baroclinic) and slow (barotropic) modes from the dispersion relation, the study offers new insights into how density stratification affects the behavior of these waves.

2. MATHEMATICAL FORMULATION

A 2D cartesian coordinate system (xy) is deployed involving a system with two fluid layers with densities ρ_1 and ρ_2 ($\rho_1 < \rho_2$) separated by a thin horizontal interface where the density undergoes a significant change (Figure 1). Given the vast depth of the ocean, the thin stratified layer can be simplified by treating it as a boundary line between two layers having individual constant density. This allows us to visualise wave propagation taking place both along the ocean's free surface and at the interface of these layers, a concept first thoroughly explained by Stokes in 1847. We define two time-harmonic velocity potentials in the upper and lower layers as $\Phi_j(x,z,t) = \phi_j(x,z)e^{-i\omega t}$ for $j = 1,2$, which satisfy the following boundary value problem (BVP) of a slighty compressible ocean

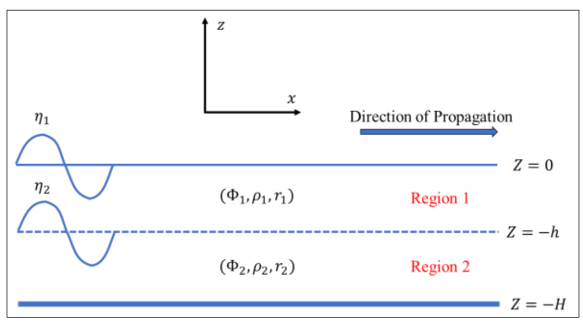

Figure 1: Schematic of physical problem

$$\nabla^2\Phi_1 = \frac{1}{c_1^2}\frac{\partial^2\Phi_1}{\partial t^2} \text{ in } -h < z < 0; \quad \nabla^2\Phi_2 = \frac{1}{c_2^2}\frac{\partial^2\Phi_2}{\partial t^2} \text{ in } -H < z < -h,$$

$$\left(\frac{\partial^2\Phi_1}{\partial t^2} + g\frac{\partial\Phi_1}{\partial z}\right)_{z=0} = 0; \ \left(\frac{\partial\Phi_2}{\partial z}\right)_{z=-H} = 0; \ \rho_2\left(\frac{\partial^2\Phi_2}{\partial t^2} + g\frac{\partial\Phi_2}{\partial z}\right)_{z=-h} = \rho_1\left(\frac{\partial^2\Phi_1}{\partial t^2} + g\frac{\partial\Phi_1}{\partial z}\right)_{z=-h},$$

where c_1, c_2 are speeds of sound in layers 1 and 2 respectively, and g is the accelaration due to gravity. Using separation of variables to solve the BVP and furthur simplying, we obtain the dispersion relation given as

$$P\omega^4 - Q\omega^2 + R = 0,$$

where $P = \frac{1}{r_1}[sr_2 \tan(ir_1 h)\tan\{ir_2(H-h)\} - r_1]$, $Q = ig[r_2\tan\{ir_2(H-h)\} + r_1\tan(ir_1 h)]$,

$R = (1-s)g^2 r_1 r_2 \tan(ir_1 h)\tan\{ir_2(H-h)\}$ and $s = {}^{\rho_1}/_{\rho_2}$.

The wavenumbers r_1 and r_2 are defined by $r_i = \sqrt{k^2 - \frac{\omega^2}{c_i^2}}$ with $\Re(r_i) \geq 0$ for $i = 1,2$. The faster (baroclinic) and the slower (barotropic) modes, denoted by ω_+ and ω_- respectively are given by

$$\omega_\pm^2 = \frac{Q \pm \sqrt{Q^2 - 4PR}}{2P}.$$

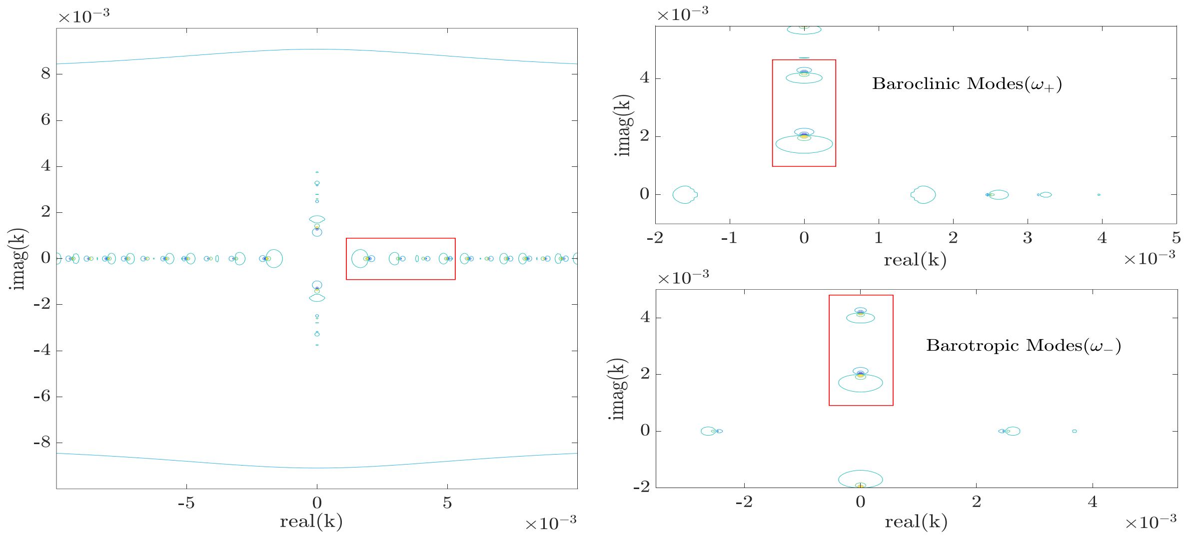

Figure 2: When water compressibility is considered, evanescent modes transform into AGW modes. The right subplots show the first four evanescent modes for an incompressible ocean. These evolve into AGW modes in the left subplot with compressibility included. Parameter values are $H = 5000m, h = 2000m, \omega = 2\pi\, rads^{-1}$, $s = 0.98, c_1 = 1450ms^{-1}, c_2 = 1500ms^{-1}, g = 9.81ms^{-2}$ throughout, unless stated otherwise.

The final forms of the potential functions are obtained to be

$\Phi_1(x,z,t) = \left\{A_1\cos(ir_1 z) - \frac{\omega a_1}{r_1}\sin(ir_1 z)\right\}e^{i(kx-\omega t)}$ & $\Phi_2(x,z,t) = A_3\cos\{ir_2(z+H)\}e^{i(kx-\omega t)}$,

where, after eliminating arbitrary constants A_1 and A_3, the amplitude ratio of the upper layer wave to the lower layer interfacial wave appears to be of the form

$$\frac{a_1}{a_2} = \frac{\cos(ir_1 h)\cos\{ir_2(H-h)\}\mathcal{M}(r_1, r_2)}{sr_2\sin\{ir_2(H-h)\}},$$

with $\mathcal{M}(r_1, r_2) = sr_2\tan\big(ir_2(H-h)\big) + r_1\tan(ir_1 h) + \frac{ig(1-s)r_1 r_2}{\omega^2}\tan\{ir_2(H-h)\}\tan(ir_1 h)$.

3. DISPERSION RELATION ANALYSIS

To study the propagation of AGW modes in a stratified ocean, the investigation seeks imaginary values of r_1 and/or r_2 that produce real values of wavenumber k for a given frequency. In theory, three possible scenarios can arise: propagation in both layers, propagation in only one of the layers, or no propagation at all. Here, however, we focus specifically on the case where propagation occurs in both layers of the ocean. In such situations we can find imaginary pairs $r_{1,2}$, such that $r_{1,2} = i\lambda_{1,2}$, which modify the relation between the vertical wavenumbers and dispersion relation:

$$\lambda_2^2 - \lambda_1^2 = \omega^2 \left(\frac{1}{c_2^2} - \frac{1}{c_1^2}\right) \quad \text{and} \quad P\omega^4 - Q\omega^2 + R = 0,$$

with $P = \frac{1}{r_1}[sr_2 \tan(r_1 h) \tan\{r_2(H-h)\} - r_1]$, $Q = g[r_2 \tan\{r_2(H-h)\} + r_1 \tan(r_1 h)]$,

$R = (s-1)g^2 r_1 r_2 \tan(r_1 h) \tan\{r_2(H-h)\}$.

The other possible existential scenarios of propagation will be touched upon during the presentation.

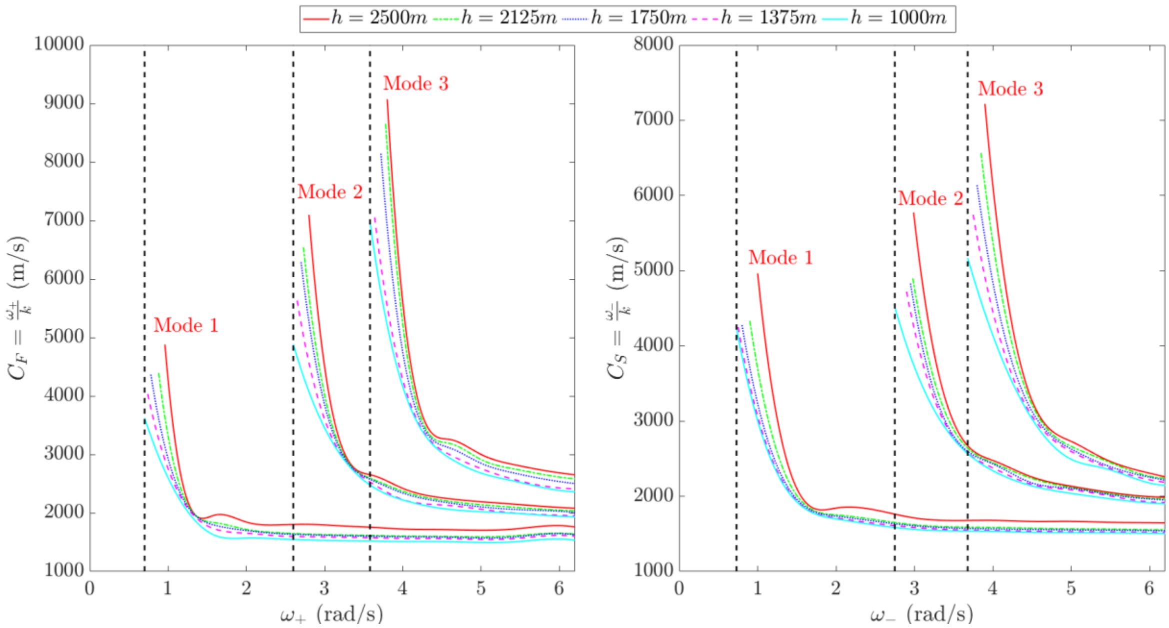

Figure 3: Variation in phase velocities $(\omega_{\pm}/k)$ of the first three fast (baroclinic) and slow (barotropic) modes against angular frequencies $(\omega_{\pm})$ are shown in left and right subplots respectively, for five choices of interface positions(h).

4. NUMERICAL RESULTS AND CONCLUSIONS

By numerically solving the dispersion relation ($H = 5000m, s = 0.98$), both types of modes are extracted. Figure 3 illustrates that AGWs persist in both layers beyond a specific wave frequency, indicated by dashed vertical lines representing the cut-off frequencies. Below these thresholds, the modes vanish. Additional results and figures will be discussed further in the oral presentation.

REFERENCES

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